SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2015

HSC Task #2

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using black or blue pen.
- Board approved calculators may be used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Leave your answers in the simplest exact form, unless otherwise stated.
- Start each **NEW** question in a separate answer booklet.

Total Marks – 100

Section I Pages 2–6

10 Marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section.

Section II Pages 7–13

90 marks

- Attempt Questions 11–16
- Allow about 1 hour and 45 minutes for this section

Examiner: *P. Parker*

Section I – Multiple Choice

10 Marks
Attempt question 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10

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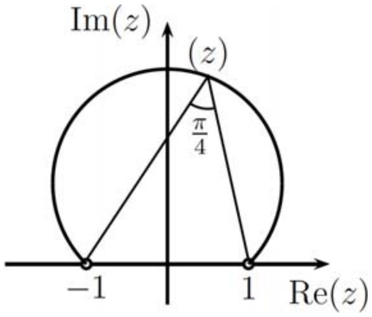
1 In the diagram below, the point z lies on a circle such that $\arg\left(\frac{z-1}{z+1}\right)=\frac{\pi}{4}$.

Which of the following equations best represents the locus of z?



(A) $|z-i|\le 2$

(B) $|z-i|=2$

(C) $|z-i|\le\sqrt{2}$

(D) $|z-i|=\sqrt{2}$

2 Consider the curve with equation $y=\left[p(x)\right]^m$, for integer m.

Which of the following statement best describes a feature of the curve of $y=\left[p(x)\right]^m$?

(A) The curve exhibits point symmetry about (0, 0).

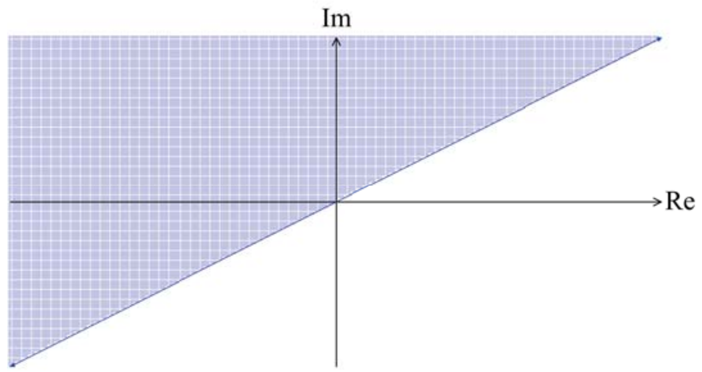
(B) The x-intercepts of $y=p(x)$ correspond to stationary points on $y=\left[p(x)\right]^m$.

(C) The curve does not exist for values for which $p(x)<0$.

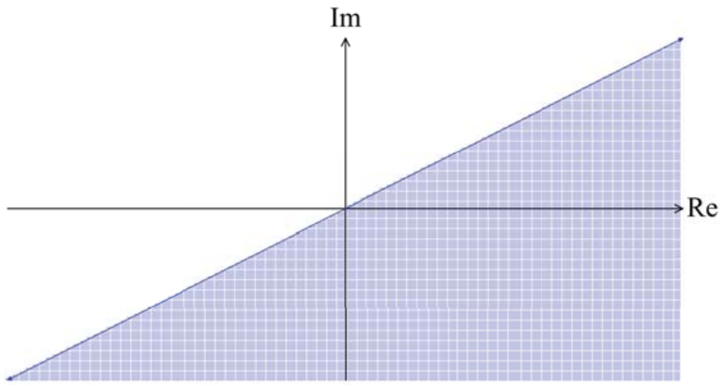
(D) The nature of the stationary points of $y=p(x)$ are preserved for $y=\left[p(x)\right]^m$.

3 Which of the following diagrams represents the locus of all points z such that $|z - i| \geq |z - 1|$?

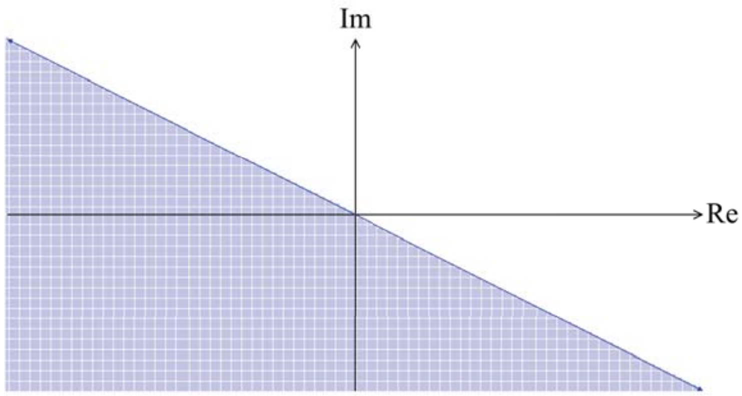
(A)



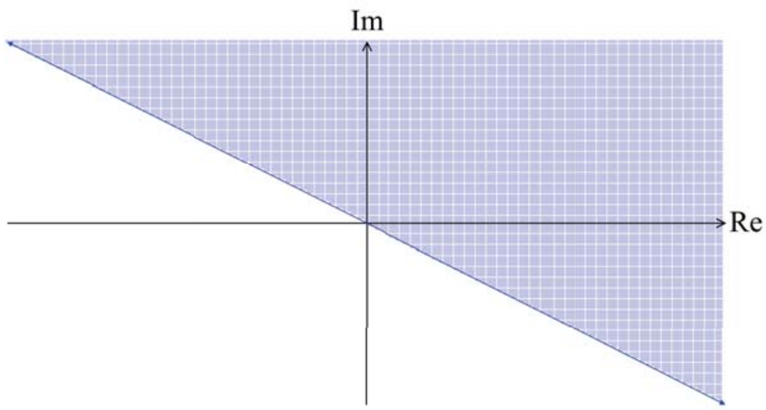
(B)



(C)



(D)



4 For the function $y = \tan^{-1}\left(e^{2x}\right)$, what is the range?

(A) $0 \le y \le \dfrac{\pi}{2}$

(B) $0 < y \le \dfrac{\pi}{2}$

(C) $0 \le y < \dfrac{\pi}{2}$

(D) $0 < y < \dfrac{\pi}{2}$

5 Consider the following two statements:

I: $\displaystyle\int_0^1 \frac{1}{1+x^{n+1}}\,dx > \int_0^1 \frac{1}{1+x^{n+2}}\,dx$ **II:** $\displaystyle\int_0^{\frac{\pi}{4}} \sqrt{\sin 2x}\,dx = \int_0^{\frac{\pi}{4}} \sqrt{\cos 2x}\,dx$

Which of these statements are correct?

(A) Neither statement.

(B) Statement I only.

(C) Statement II only.

(D) Both statements.

6 Which of the following would best describe how the graph of the function $y = 2^{x^2-4x+3}$ can be obtained from the graph of $y = 2^{x^2}$?

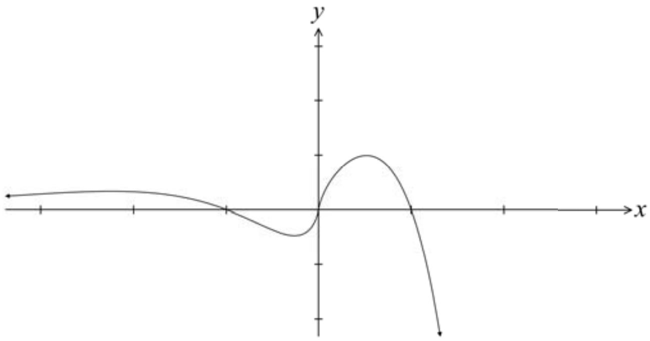
(A) A stretch parallel to the *y*-axis followed by a translation parallel to the *y*-axis.

(B) A translation parallel to the *x*-axis followed by a stretch parallel to the *y*-axis.

(C) A stretch parallel to the *x*-axis followed by a translation parallel to the *x*-axis.

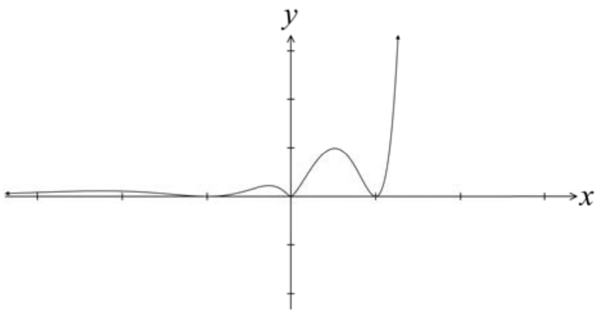
(D) A translation parallel to the *y*-axis followed by a stretch parallel to the *x*-axis.

7 The diagram below shows the graph of the function $y = f(x)$.

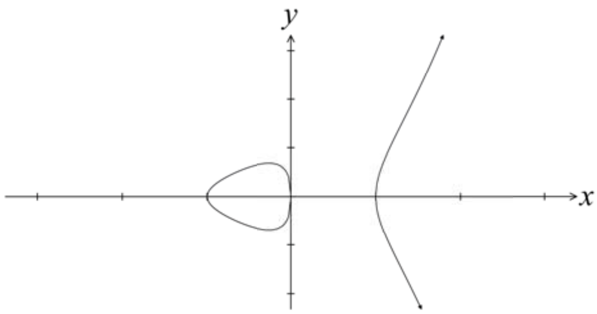


Which of the following could be the graph of $y = \sqrt{f(x)}$?

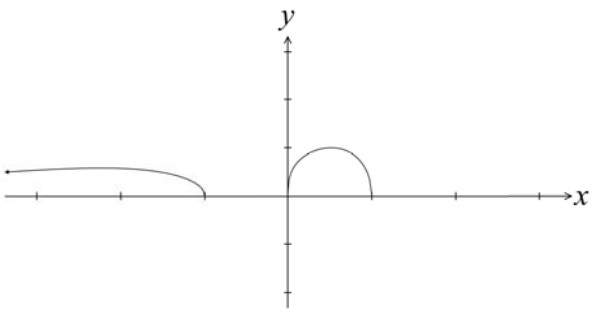
(A)



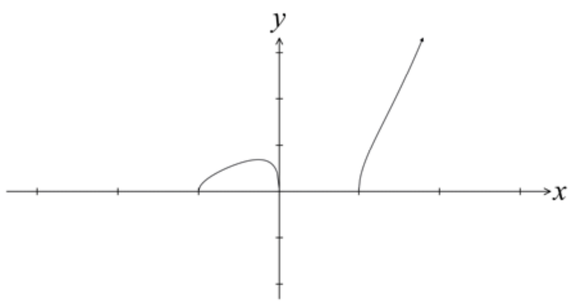
(B)



(C)



(D)



8 Which of the following could be the derivative of $\ln[(x+y)^2]$ with respect to x, where y represents a differentiable function of x.

(A) $\dfrac{2(1+\frac{dy}{dx})}{x+y}$

(B) $2(x+y)(1+\frac{dy}{dx})$

(C) $\dfrac{2(1+\frac{dy}{dx})}{(x+y)^2}$

(D) $\dfrac{1+\frac{dy}{dx}}{x+y}$

9 Which of the following is the same as $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^3 2x\, dx$?

(A) $\displaystyle\int_0^{\frac{\sqrt{3}}{2}} (1-u^2)\, du$

(B) $\displaystyle\int_0^{\frac{\sqrt{3}}{2}} (u^2-1)\, du$

(C) $\displaystyle\frac{1}{2}\int_0^{\frac{\sqrt{3}}{2}} (1-u^2)\, du$

(D) $\displaystyle\frac{1}{2}\int_0^{\frac{\sqrt{3}}{2}} (u^2-1)\, du$

10 Fifteen identical boxes are being sent to five distinct people.
How many different ways can the boxes get distributed if it is possible for people to get no boxes (e.g. all the boxes get lost)?

(A) $\dfrac{15!}{5!10!}$

(B) $\dfrac{19!}{4!15!}$

(C) $\dfrac{20!}{5!15!}$

(D) $\dfrac{21!}{5!16!}$

Section II

90 marks
Attempt Questions 10–16
Allow about 1 hour and 45 minutes for this section

Answer each question in a NEW writing booklet. Extra pages are available

In Questions 10–16, your responses **should include** relevant mathematical reasoning and/or calculations.

Question 11 (15 Marks) Start a NEW Writing Booklet

(a) Use the substitution $u = 1 + 3\tan x$ to find the exact value of $\int_0^{\frac{\pi}{4}} \frac{\sqrt{1+3\tan x}}{\cos^2 x}\,dx$ **3**

(b) Find $\int x\tan(x^2)\,dx$. **2**

(c) Find $\int \frac{2x-9}{2(x-3)\sqrt{x-3}}\,dx$ **2**

(d) Let $f(x) = \frac{6+6x}{(2-x)(2+x^2)}$.

(i) Express $f(x)$ in the form $\frac{A}{2-x} + \frac{Bx+C}{2+x^2}$ **2**

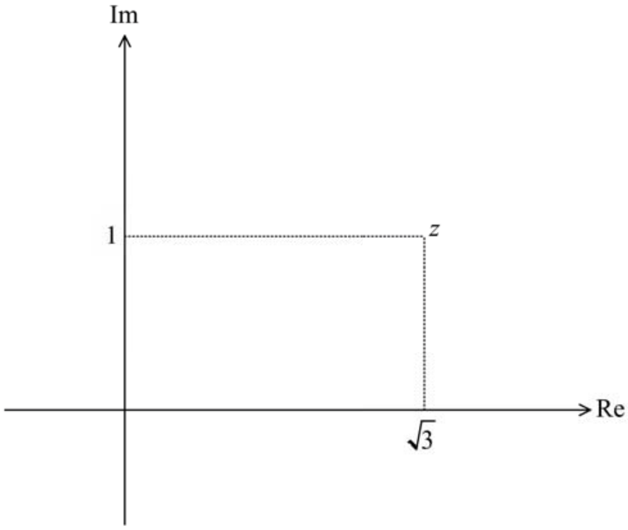
(ii) Show that $\int_{-1}^{1} f(x)\,dx = 3\ln 3$ **3**

(e) Find $\int x\sec^4 x\tan x\,dx$ **3**

Question 12 (15 Marks) Start a NEW Writing Booklet

(a) Sketch the region where the inequalities $-\frac{\pi}{2} \leq \arg(z-1-2i) \leq \frac{\pi}{4}$, and $|z| \leq \sqrt{5}$ both hold. **3**

(b) Suppose that $z = \sqrt{3} + i$ and $\omega = (\text{cis}\theta)z$ where $-\pi < \theta \leq \pi$.



(i) Find the argument of z. **1**

(ii) Find the value of θ if ω is purely imaginary and $\text{Im}\,\omega > 0$. **2**

(iii) Find the value of $\arg(z+\omega)$ if ω is purely imaginary and $\text{Im}\,\omega > 0$. **2**

(c) Let two non-zero complex numbers be z_1 and z_2.
Let θ be the angle between the straight lines joining 0 to z_1 and 0 to z_2.

(i) One possible expression for θ is $\theta = \arg z_1 - \arg z_2$. **1**
Write down another possible expression for θ in terms of $\arg z_1$ and $\arg z_2$.

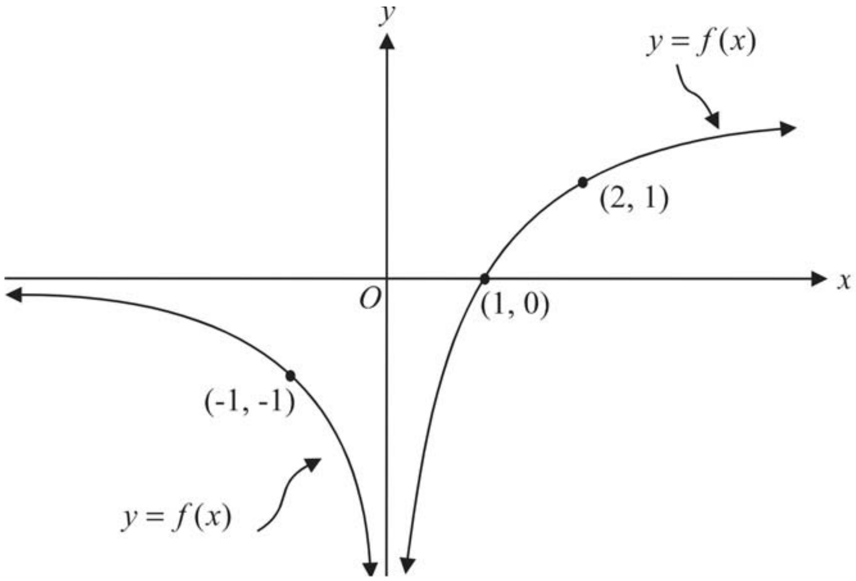
(ii) Hence prove that $\text{Im}(z_1\overline{z}_2) = \pm|z_1||z_2|\sin\theta$ **2**

(iii) Hence find the area of the triangle whose vertices are 0, 1 + 3i and –3 + 2i. **2**

(d) The polynomial equation $x^3 - 3x^2 - x + 2 = 0$ has roots α, β and γ. **2**
Find the polynomial equation with roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$.

Question 13 (15 Marks) Start a NEW Writing Booklet

(a) The function f is a discontinuous function.
The diagram below shows the graph of $y = f(x)$.



Draw large (half page), separate sketches of each of the following:

(i) $y = f\left(\dfrac{x}{2}\right)$ **1**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = f(2-x)$ **2**

(iv) $y = \tan^{-1} f\left(x\right)$ **2**

(b) The equation of a curve is $2x^2 + 3xy + y^2 = 3$. **3**
Find the equation of the tangent at the point (2, –1).

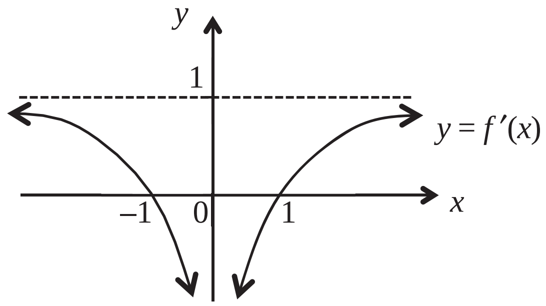
(c) A sequence of numbers T_n, for integers $n \geq 1$, is defined below. **5**

$$T_1 = 2,\ T_2 = -4 \text{ and } T_n = 2T_{n-1} - 4T_{n-2} \text{ for } n \geq 3.$$

Use mathematical induction to show that $T_n = \left(1+\sqrt{3}i\right)^n + \left(1-\sqrt{3}i\right)^n$ for $n \geq 1$.

Question 14 (15 Marks)   Start a NEW Writing Booklet

(a) The graph of $y = f'(x)$ is drawn below. **3**
Given that $f(1) = 2$ and $f(-1) = -2$, draw a sketch of $y = f(x)$.
Include any asymptotes if necessary.



(b) Let a and b be real numbers. Consider the cubic equation

$$x^3 + 2bx^2 - a^2x - b^2 = 0 \qquad (*)$$

(i) Show that if $x = 1$ is a solution of (*) then $1-\sqrt{2} \le b \le 1+\sqrt{2}$ **2**

(ii) Show that there is no value of b for which $x = 1$ is a repeated root of (*). **2**

(c) Find the six solutions of the equation $\sin\left(2\cos^{-1}\left(\cot\left(2\tan^{-1}x\right)\right)\right) = 0$. **4**

(d) Rekrap is selling raffle tickets for \$1 per ticket. In the queue for tickets, there are m people each with a single \$1 coin and n people with a single \$2 coin.
Each person in the queue wants to buy a single raffle ticket and each arrangement of people in the queue is equally likely to occur.
Initially Rekrap has no coins and a large supply of tickets. Rekrap stops selling tickets if he cannot give the required change.

(i) In the case $n = 1$ and $m \ge 1$, find the probability that Rekrap is able to sell one ticket to each person in the queue. **1**

(ii) By considering the first three people in the queue, show that the probability that Rekrap is able to sell one ticket to each person in the queue in the case $n = 2$ and $m \ge 2$ is given by **3**

$$\frac{m-1}{m+1}$$

Question 15 (15 Marks) Start a NEW Writing Booklet

(a) The polynomial $P(z)$ has equation $P(z) = z^4 - 2z^3 - z^2 + 2z + 10$. **3**

Given that $z - 2 + i$ is a factor of $P(z)$, express $P(z)$ as a product of two quadratic factors with real coefficients

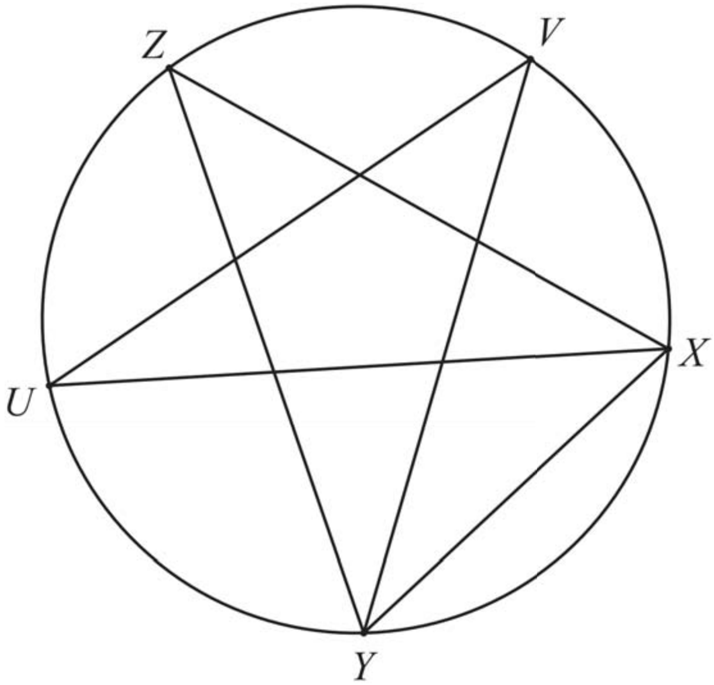
(b) A particular accounting firm has 15 CDs in a box. 12 of them have data on them and 3 of them are blank. **2**

One of the staff members obtains a CD, at random, from the box and checks to see if there is any data on it and does not place it back in the box after verifying the actual contents of the CD.

Find the probability that the 13^{th} disk checked will be the 10^{th} disk that contains data.

(c) In the diagram below, the points X, Y, and Z lie on a circle. The chord XY is a fixed chord of the circle and Z is any point such that XZY is a major arc. **3**

The chord UX is a bisector of $\angle ZXY$ and chord VY is a bisector of $\angle ZYX$.



Prove that UV is a chord of constant length, for any point Z on the major arc XZY.

Question 15 continues on page 12

Question 15 (continued)

(d) (i) Let $z = r(\cos\theta + i\sin\theta)$. **1**

Prove that $z - \overline{z} = 2ir\sin\theta$.

(ii) Let m and n be positive integers.

When $x^m(1-x)^n$ is divided by $1+x^2$ the remainder is $ax+b$.

(1) By writing $x^m(1-x)^n$ in the form **3**

$x^m(1-x)^n = A(x)Q(x) + R(x)$

show that $2ai = i^m(1-i)^n - (-i)^m(1+i)^n$

(2) Hence, or otherwise, show that $a = \left(\sqrt{2}\right)^n \sin\frac{(2m-n)\pi}{4}$ **3**

End of Question 15

Please turn over

Question 16 (15 Marks) Start a NEW Writing Booklet

(a) Let $I = \int_0^a \frac{\cos x}{\sin x + \cos x}\, dx$ and $J = \int_0^a \frac{\sin x}{\sin x + \cos x}\, dx$, where $0 \le a \le \frac{3\pi}{4}$.

Show that $2I = a + \ln(\sin a + \cos a)$ **3**

(b) (i) Use the substitution $x = \frac{1}{t^2 - 1}$, where $t > 1$, to show that, for $x > 0$ **4**

$$\int \frac{1}{\sqrt{x(x+1)}}\, dx = 2\ln\left(\sqrt{x} + \sqrt{x+1}\right) + C$$

(ii) Hence show that $\int_{\frac{1}{8}}^{\frac{9}{16}} \left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+1}}\right)^2 dx = 2\ln\frac{5}{4}$. **2**

(c) For any given function f, let $I = \int \left[f'(x)\right]^2 \left[f(x)\right]^n dx$, where n is a positive integer.

Also, $f(x)$ also satisifies $f''(x) = k\, f'(x)\, f(x)$ for some constant k.

(i) By using integration by parts, or otherwise, show that **3**

$$I = \frac{f'(x)\left[f(x)\right]^{n+1}}{n+1} - \frac{k\left[f(x)\right]^{n+3}}{(n+1)(n+3)} + C \text{, for some constant } C.$$

(ii) Hence, or otherwise, find $\int \sec^2 x(\sec x + \tan x)^6\, dx$. **3**

End of paper

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STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

SYDNEY BOYS HIGH SCHOOL
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2015

HSC Task #2

Mathematics Extension 2

Suggested Solutions & Markers' Comments

QUESTION	Marker
1 – 10	–
11	PB
12	PB
13	AMG
14	AF
15	AMG
16	AF

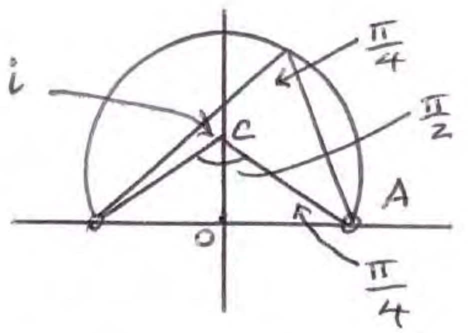
Multiple Choice Answers

1. D
2. B
3. B
4. D
5. C
6. B
7. C
8. A
9. D
10. C

Solutions for Multiple Choice Y12 Ext 2 Task 2 2015

The mean score for this question was 6.32/10

Q1



Angle at centre $= \frac{\pi}{2}$

$\therefore$ Angle at A is $\frac{\pi}{4}$

$\therefore OC = OA = 1$

$\therefore C$ is i

$\therefore AC = \sqrt{2}$

$\therefore |z - i| = \sqrt{2}$

D

A	1
B	19
C	4
D	92

Q2

- Point symmetry $\Rightarrow$ odd function
- $p(x) < 0$ causes no problems for n an integer
- Consider $y = (x^3)^2$

  For $y = x^3$, stationary point of inflexion at $x = 0$

  For $y = x^6$, minimum turning point at $x = 0$

  $\therefore$ Nature not preserved.
- x intercepts of $p(x)$ are stationary points for $(p(x))^n$

B

A	7
B	66
C	2
D	41

Q3

A is $|z - i| \le |z - 1|$

B is $|z - i| \ge |z - 1|$

C is $|z + i| \le |z - 1|$

D is $|z + i| \ge |z - 1|$

B

A	32
B	74
C	4
D	6

Q4

$0 < e^{2x}$

$\therefore\ 0 < \tan^{-1}(e^{2x}) < \frac{\pi}{2}$

D

A	5
B	8
C	11
D	92

Q5

5. $x^{n+1} > x^{n+2}$ if $0 < x < 1$

$\therefore 1 + x^{n+1} > 1 + x^{n+2}$

$\therefore \frac{1}{1+x^{n+1}} < \frac{1}{1+x^{n+2}}$

$\therefore \int_0^1 \frac{dx}{1+x^{n+1}} < \int_0^1 \frac{dx}{1+x^{n+2}}$

$\therefore$ I is false

$\int_0^{\pi/4} \sqrt{\sin 2x}\, dx$ Let $u = \frac{\pi}{4} - x$

$\therefore du = -dx$

$= \int_{\frac{\pi}{4}}^{0} \sqrt{\sin(\frac{\pi}{2} - 2u)}\,(-du)$ If $x = 0,\ u = \frac{\pi}{4}$

$x = \frac{\pi}{4},\ u = 0$

$= \int_0^{\frac{\pi}{4}} \sqrt{\cos 2u}\, du$

$= \int_0^{\pi/4} \sqrt{\cos 2x}\, dx$

$\therefore$ II is true

C

A	21
B	24
C	46
D	25

Q6

6. $y = 2^{x^2-4x+3}$

$= 2^{x^2-4x+4-1}$

$= 2^{(x-2)^2} \cdot 2^{-1}$

$= \frac{1}{2} \cdot 2^{(x-2)^2}$

$\therefore\ y = 2^{x^2-4x+3}$

involves a translation of 2 units to the right and then shrinking

$\therefore$ B

A	19
B	38
C	37
D	21

Q7

7. $y = \sqrt{f(x)}$ is not defined for $-1 < x < 0$ or for $x > 1$

(assuming marking are 1 unit apart)

C is the only diagram where the function is not defined in these areas.

A	0
B	1
C	115
D	0

Q8

$$\frac{d}{dx}\left(\ln (x+y)^2\right)$$

$$= \frac{d}{dx}\left(2 \ln(x+y)\right)$$

$$= 2 \cdot \frac{1}{x+y} \cdot (1+y')$$

A

A	101
B	2
C	13
D	0

Q9

$$\int_{\frac{\pi}{3}}^{\pi/2} \cos^3 2x \, dx$$

$$= \int_{\pi/3}^{\pi/2} \cos^2 2x . \cos 2x \, dx$$

$$= \int_{\pi/3}^{\pi/2} (1 - \sin^2 2x) . \cos 2x \, dx$$

Let $u = \sin 2x$

$\therefore du = 2 \cos 2x \, dx$

$$= \int_{\frac{\sqrt{3}}{2}}^{0} (1 - u^2) . \tfrac{1}{2} du$$

if $x = \frac{\pi}{3}$ $u = \frac{\sqrt{3}}{2}$

$x = \frac{\pi}{2}$ $u = 0$

$$= \frac{1}{2} \int_0^{\sqrt{3}/2} (u^2 - 1) \, du$$

D

A	4
B	7
C	48
D	57

Q10

This situation can be thought of as having 15 items

× × × × ... ×

and inserting 5 separators.
There are 16 positions for the first separator, 17 for the second ... 20 for the fifth.

∴ There are 16×17× ... ×20 ways to insert the separators.
As the separators are indistinguishable divide by 5!

∴ Number of ways $= \frac{20!}{5!\,15!}$

OR

15 boxes and 5 separators.

∴ 20! arrangements.
We divide by 15! as the boxes are indistinguishable and by 5! as the separators are indistinguishable

∴ $\frac{20!}{5!\,15!}$ ways

C

A	13
B	29
C	52
D	21

(x 2)

QUESTION 11.

(a) $\int_0^{\frac{\pi}{4}} \frac{\sqrt{1+3\tan x}}{\cos^2 x}\,dx$

let
$u = 1+3\tan x$
$du = 3\sec^2 x\,dx$
$= \frac{3\,dx}{\cos^2 x}$

$= \frac{1}{3}\int_1^4 u^{\frac{1}{2}}\,du$

$= \frac{2}{9}\left[u^{3/2}\right]_1^4$

$= \frac{2}{9}(8-1)$

$= \frac{14}{9}$

COMMENT

Well done!

(b) $\int x\tan(x^2)\,dx$

let $u = x^2$
$du = 2x\,dx$

$= \frac{1}{2}\int \tan u \,.\, du$

$= \frac{1}{2}\int \frac{\sin u}{\cos u}\,.\,du$

$= -\frac{1}{2}\ln(\cos u) + c$

$= -\frac{1}{2}\ln(\cos(x^2)) + c.$

COMMENT

Those who saw this as a substitution question generally obtained full marks.

Q11 (CONTD)

(c) $\displaystyle\int \frac{2x-9}{2(x-3)\sqrt{x-3}}\,dx = \int \frac{2x-6-3}{2(x-3)\sqrt{x-3}}\cdot dx.$

$$= \int \frac{dx}{\sqrt{x-3}} - \frac{3}{2}\int \frac{dx.}{(x-3)\sqrt{x-3}}$$

$$= \int (x-3)^{-\frac{1}{2}}dx - \frac{3}{2}\int (x-3)^{-3/2}\cdot dx.$$

$$= 2(x-3)^{\frac{1}{2}} + 3(x-3)^{-\frac{1}{2}} + c$$

$$= 2\sqrt{x-3} + \frac{3}{\sqrt{x-3.}} + c$$

OR. $\boxed{\dfrac{2x-3}{\sqrt{x-3}} + c}$

COMMENT most obtained full marks.

(d). (i) Let $6+6x \equiv A(2+x^2) + (2-x)(Bx+c)$

if $x=2$ $\quad 6+12 = 6A.$

$\therefore \boxed{A=3}$

if $x=0$ $\quad 6 = 2A + 2c$

$\therefore \boxed{c=0}$

if $x=1$ $\quad 12 = 3A+B+c$

$12 = 9+B+0$

$\therefore \boxed{B=3}$

Q11 (d) (i) (CONTD)

$$\therefore \frac{6+6x}{(2-x)(2+x^2)} = \frac{3}{2-x} + \frac{3x}{2+x^2}$$

Hence. $$\int_{-1}^{1} \frac{6+6x}{(2-x)(2+x^2)}\,dx = \int_{-1}^{1} \frac{3\,dx}{2-x} + \int_{-1}^{1} \frac{3x\,dx}{2+x^2}$$

$$= -3\Big[\ln[2-x]\Big]_{-1}^{1} + 0$$

$$= -3\left[\ln 1 - \ln 3\right]$$

$$= \boxed{3\ln 3}$$

COMMENT * need to recognise that $\frac{3x}{2+x^2}$ is an odd function.

* Alternatively $\int_{-1}^{1} \frac{3x\,dx}{2+x^2}$

$$= \frac{3}{2}\Big[\ln(2+x^2)\Big]_{-1}^{1}$$

$$= \frac{3}{2}\left[\ln 3 - \ln 3\right]$$

$$= 0.$$

* Question was well done.

Q11 (CONTD)

(e). $\int x \sec^4 x \tan x \, dx = \int x \sec^3 x . \sec x \tan x \, dx$

$= \int x . \frac{d}{dx}\left(\frac{\sec^4 x}{4}\right) dx$

$= \frac{1}{4} x \sec^4 x - \frac{1}{4}\int \sec^4 x \, dx$

$= \frac{1}{4} x \sec^4 x - \frac{1}{4}\int (1+\tan^2 x)\sec^2 x \, dx$

$= \frac{1}{4} x \sec^4 x - \frac{1}{4}\int \sec^2 x \, dx - \frac{1}{4}\int \sec^2 x \tan^2 x \, dx$

$= \frac{1}{4} x \sec^4 x - \frac{1}{4}\tan x - \frac{1}{4} \times \frac{1}{3}\tan^3 x + c$

$= \frac{1}{4} x \sec^4 x - \frac{1}{4}\tan x - \frac{1}{12}\tan^3 x + c$

COMMENT * An alternative approach was to let $\int x \sec^4 x \tan x \, dx$

$= \int x (\tan^2 x + 1) \sec^2 x \tan x \, dx$

$= \int x \tan^3 x \sec^2 x \, dx + \int x \sec^2 x \tan x \, dx$

this led to

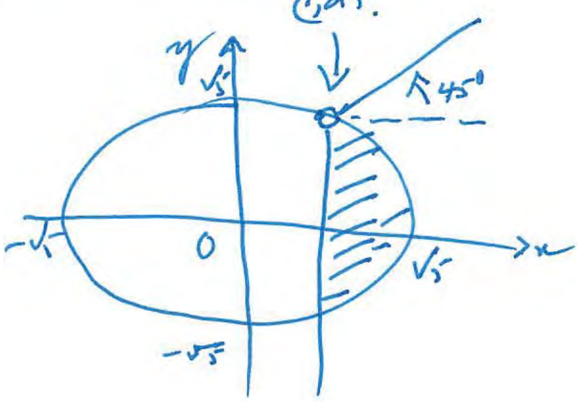
$\frac{1}{4} x \tan^4 x + \frac{1}{2} x \tan^2 x - \frac{\tan x}{4} + \frac{x}{4} - \frac{1}{12}\tan^3 x + c$

* This question proved too difficult for most students, given time constraints.

(× 2)

QUESTION 12

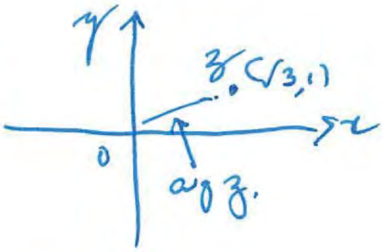
(a)



COMMENT

A significant number failed to recognise that (1,2) lies on the circle.

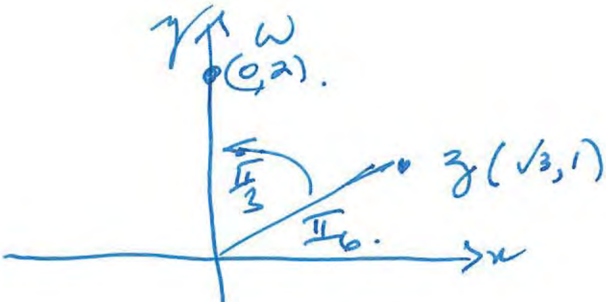
(b) (i)



$\arg z = \tan^{-1}\frac{1}{\sqrt{3}}$

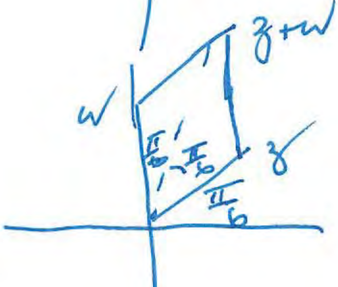
$= \frac{\pi}{6}$

(ii)



$\frac{\pi}{3}$

(iii)



Since a rhombus

$\arg(z+w) = \frac{\pi}{6} + \frac{\pi}{6}$

$= \frac{\pi}{3}$

OR. $z+w = \sqrt{3}+i + 0+2i$

$= \sqrt{3}+3i$

$\therefore \arg(z+w) = \tan^{-1}\frac{3}{\sqrt{3}}$

$= \tan^{-1}\sqrt{3}$

$= \frac{\pi}{3}$.

Q12 (CONTD)

COMMENT (b)

Parts (i) & (ii) were well done.

(iii) many thought that

$\arg(z+w) = \arg z + \arg w$.

which is a serious mistake!

(c) (i) θ could be $\arg z_1 - \arg z_2$

OR $\arg z_2 - \arg z_1$

COMMENT $\arg \frac{z_2}{z_1}$ was accepted although the question asked for the answer in terms of $\arg z_1$ and $\arg z_2$.

(ii) let $\theta_1 = \arg z_1$ & $\theta_2 = \arg z_2$.

$\therefore \theta = \pm(\theta_1 - \theta_2)$ OR $\theta_1 - \theta_2 = \pm\theta$ (A)

now $z_1 \bar{z}_2 = |z_1| \operatorname{cis} \theta_1 \times |z_2| \operatorname{cis} -\theta_2$

$= |z_1||z_2| \operatorname{cis}(\theta_1 - \theta_2)$

$\therefore \operatorname{Im}(z_1 \bar{z}_2) = |z_1||z_2| \sin(\theta_1 - \theta_2)$

$= \pm |z_1||z_2| \sin\theta$ from (A).

(iii) Area of triangle is $\frac{1}{2}|z_1||z_2| \sin\theta$.

$= \frac{1}{2} \operatorname{Im}(z_1 \bar{z}_2)$

Q1(c)(iv) (CONTD)

now $(1+3i)(\overline{-3+2i}) = (1+3i)(-3-2i)$.

$= 3 - 11i$

& $(-3+2i)(\overline{1+3i}) = (-3+2i)(1-3i)$

$= 3 + 11i$.

$\therefore \operatorname{Im}(z_1 \overline{z_2})$ is ± 11.

$\therefore$ area of triangle is $\frac{1}{2}|-11|$

$= \frac{11}{2}$ u²

COMMENT (i) Very few answered this correctly. Most simply answered as $\arg \frac{z_1}{z_2}$, which is not what was required.

(ii) An understanding of (i) was needed to get $\pm |z_1||z_2| \sin\theta$.

Very few full marks.

(iii) Question asked for "HENCE" not "HENCE OR OTHERWISE".

(d) let $x = \frac{1}{x}$ $\therefore$ $x^3 - 3x^2 - x + 2 = 0$

becomes $\left(\frac{1}{x}\right)^3 - 3\left(\frac{1}{x}\right)^2 - \left(\frac{1}{x}\right) + 2 = 0$

i.e. $2x^3 - x^2 - 3x + 1 = 0$

COMMENT Most obtained full marks.

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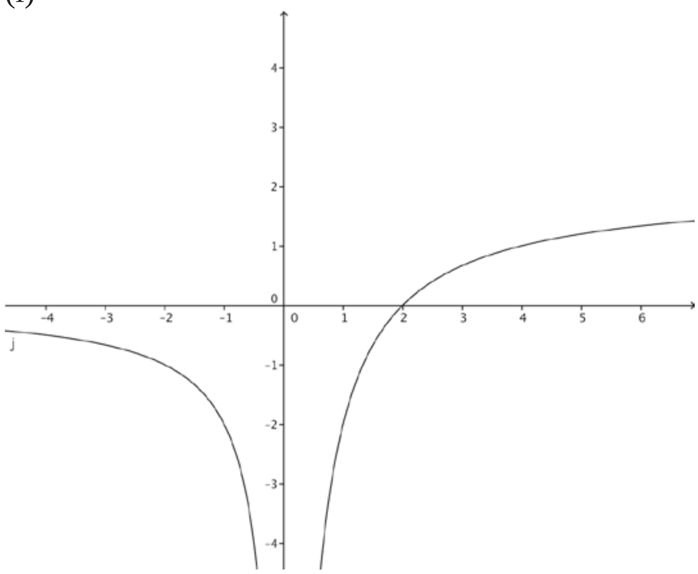
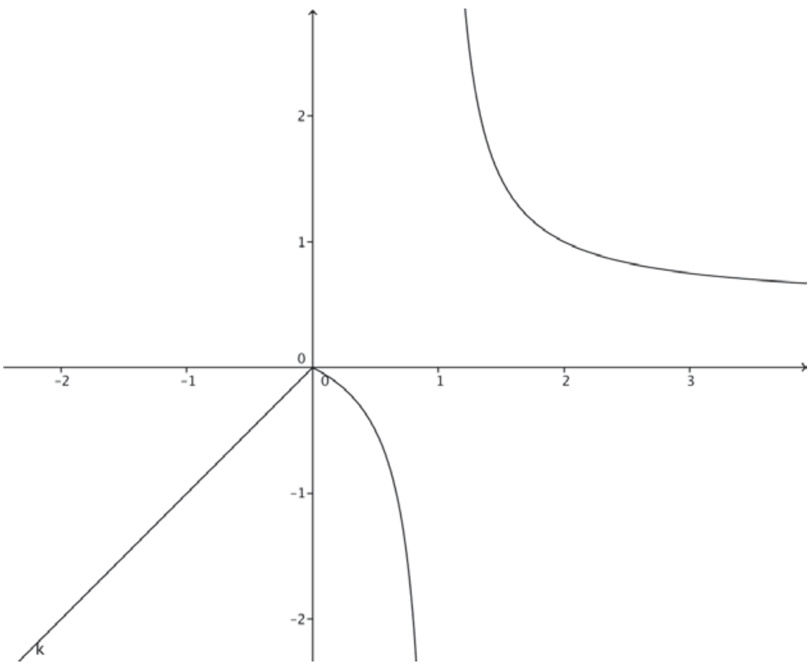
Question 13

(a) (i)



Comments: The main error was to graph $y = f(2x)$.

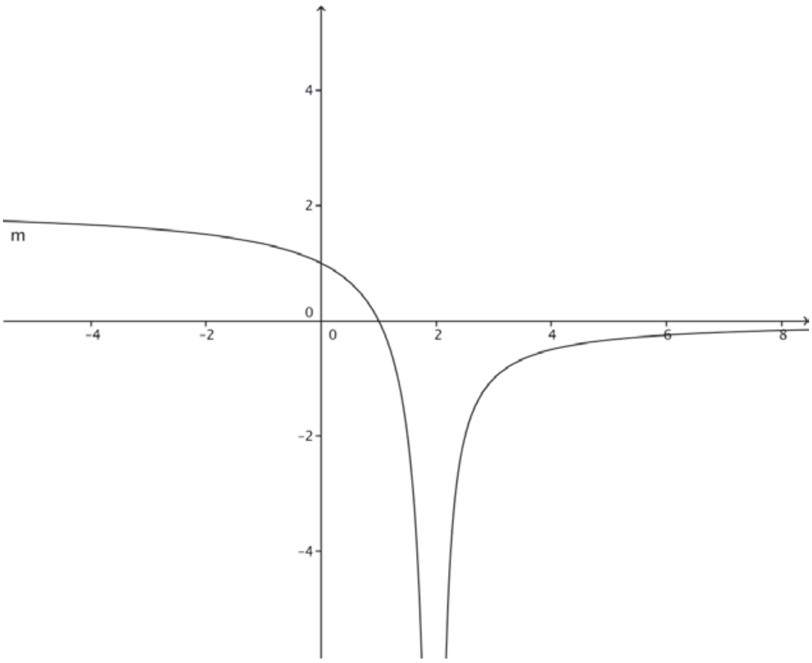
(ii)



Note: The origin is an open point, not in the solution.

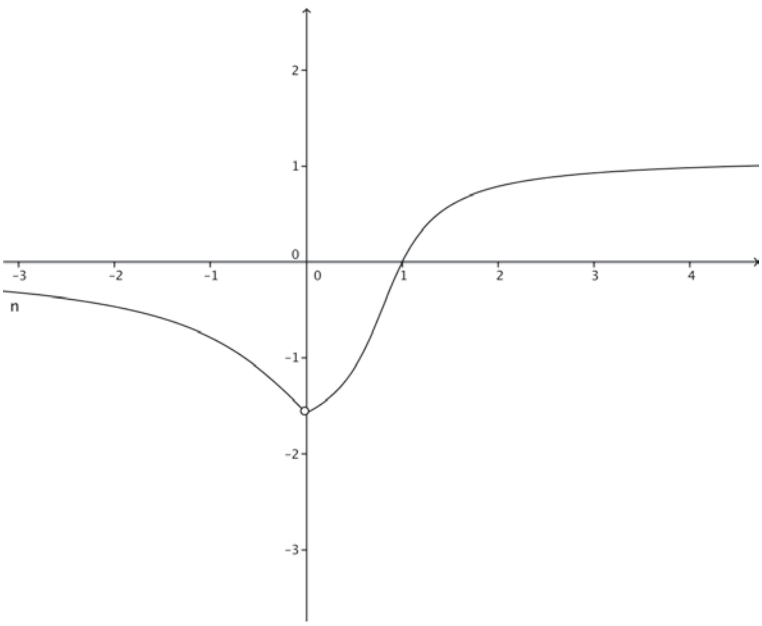
Comments: Many failed to indicate the origin as open, whilst others had the middle part of the curve above the axis.

(iii)



Comments: This was surprisingly poorly answered, by about half the candidates. They flipped y values, and had the asymptote at $x = -2$.

(iv)



Note: Asymptote at $y = \frac{\pi}{2}$, the open point is $\left(0, \frac{\pi}{2}\right)$.

Comments: Surprisingly well answered.

(b) $2x^2 + 3xy + y^2 = 3$

Differentiating implicitly

$4x + 3xy' + 3y + 2yy' = 0$

$y' = \frac{-4x - 3y}{3x + 2y}$

At $(2, -1)$

$y' = -\frac{5}{4}$

Hence tangent:

$y - (-1) = -\frac{5}{4}(x - 2)$

$5x + 4y - 6 = 0$

Comment: Many were careless in their differentiation, and some failed to give the equation of the tangent.

(c) [Excised from paper.]

$T_1 = 2,\ T_2 = -4,\ T_n = 2T_{n-1} - 4T_{n-2}$

$p(n):\ T_n = \left(1 + i\sqrt{3}\right)^n + \left(1 - i\sqrt{3}\right)^n$

$p(1):\quad LHS = 2$

$RHS = \left(1 + i\sqrt{3}\right)^1 + \left(1 - i\sqrt{3}\right)^1$

$= 2 = LHS$

$p(2):\quad LHS = -4$

$RHS = \left(1 + i\sqrt{3}\right)^2 + \left(1 - i\sqrt{3}\right)^2$

$= -4 = LHS$

Thus the proposition is true for $n = 1, \quad n = 2$.

$p(k)$: Assume $T_k = \left(1 + i\sqrt{3}\right)^k + \left(1 - i\sqrt{3}\right)^k$ is true for all positive integers less than or equal to *k*.

Required to Prove that this implies $p(k+1)$ is true.

ie $\quad T_{k+1} = \left(1 + i\sqrt{3}\right)^{k+1} + \left(1 - i\sqrt{3}\right)^{k+1}$

$LHS = 2T_k - 4T_{k-1}$

$$= 2\left[\left(1+i\sqrt{3}\right)^{k} + \left(1-i\sqrt{3}\right)^{k}\right] - 4\left[\left(1+i\sqrt{3}\right)^{k-1} + \left(1-i\sqrt{3}\right)^{k-1}\right]$$

$$= 2\left[\left(1+i\sqrt{3}\right)\left(1+i\sqrt{3}\right)^{k-1} + \left(1-i\sqrt{3}\right)\left(1-i\sqrt{3}\right)^{k-1}\right] - 4\left[\left(1+i\sqrt{3}\right)^{k-1} + \left(1-i\sqrt{3}\right)^{k-1}\right]$$

$$= \left(1+i\sqrt{3}\right)^{k-1}\left[2\left(1+i\sqrt{3}\right) - 4\right] + \left(1-i\sqrt{3}\right)^{k-1}\left[2\left(1-i\sqrt{3}\right) - 4\right]$$

$$= \left(1+i\sqrt{3}\right)^{k-1}\left[2\left(1+i\sqrt{3}\right) + \left(1+i\sqrt{3}\right)^{2} + \left(1-i\sqrt{3}\right)^{2}\right] +$$

$$\left(1-i\sqrt{3}\right)^{k-1}\left[2\left(1-i\sqrt{3}\right) + \left(1+i\sqrt{3}\right)^{2} + \left(1-i\sqrt{3}\right)^{2}\right]$$

$$== \left(1+i\sqrt{3}\right)^{k+1} + \left(1-i\sqrt{3}\right)^{k+1} + 2\left[\left(1+i\sqrt{3}\right)^{k} + \left(1-i\sqrt{3}\right)^{k}\right] + \left(1+i\sqrt{3}\right)^{k-1}\left(1-i\sqrt{3}\right)^{2} +$$

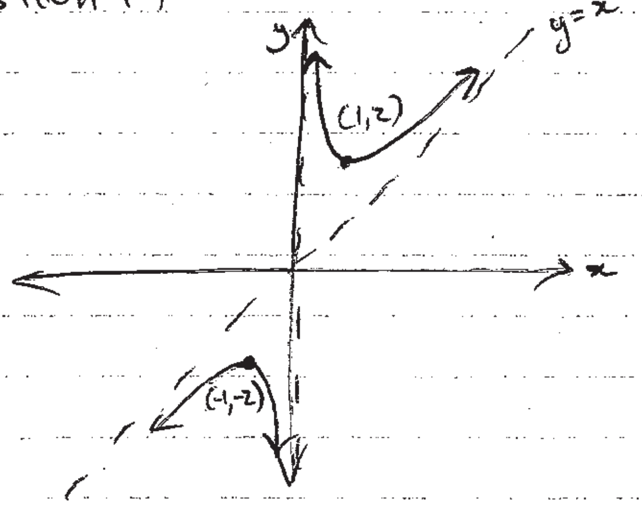
$$\left(1-i\sqrt{3}\right)^{k-1}\left(1+i\sqrt{3}\right)^{2}$$

$$== \left(1+i\sqrt{3}\right)^{k+1} + \left(1-i\sqrt{3}\right)^{k+1} + 2\left[\left(1+i\sqrt{3}\right)^{k} + \left(1-i\sqrt{3}\right)^{k}\right] - 2\left(1+i\sqrt{3}\right)^{k} - 2\left(1-i\sqrt{3}\right)^{k}$$

Thus $\boldsymbol{p(k+1)}$ is true if the proposition is true for $n = k$, and for all positive integers less than *k*. Hence the proposition is true for all positive *n*.

Question 14

a)



COMMENT:

Successful students considered the asymptotes and the nature of the stationary points.

b) i) Let $P(x) = x^3 + 2bx^2 - a^2x - b^2$ where a & b are real

$P(1) = (1)^3 + 2b(1)^2 - a^2(1) - b^2 = 0$

$1 + 2b - a^2 - b^2 = 0$

$1 + 2b - b^2 = a^2$ ——— ①

$1 + 2b - b^2 \geq 0$, since a is real

$b^2 - 2b + 1 \leq 1 + 1$

$(b-1)^2 \leq 2$

$-\sqrt{2} \leq b - 1 \leq \sqrt{2}$

$1 - \sqrt{2} \leq b \leq 1 + \sqrt{2}$

ii) $P'(x) = 3x^2 + 4bx - a^2$

$P'(1) = 3(1)^2 + 4b(1) - a^2 = 0$

$3 + 4b = a^2$ ——— ②

sub ① into ②

$3 + 4b = 1 + 2b - b^2$

$b^2 + 2b + 2 = 0$

$b^2 + 2b + 1 = -1$

$(b+1)^2 = -1$

$\therefore$ No real value of b for which $x = 1$ is a repeated root

COMMENT:

(i) Students had trouble in getting an inequality from the equation

(ii) Many students failed to link a^2 with what was found in previous part.

c) METHOD 1

$$\sin\left(2\cos^{-1}\left(\cot\left(2\tan^{-1}x\right)\right)\right) = 0$$

$$2\cos^{-1}\left(\cot(2\tan^{-1}x)\right) = 0, \pi, 2\pi, \ldots$$

$$\cos^{-1}\left(\cot(2\tan^{-1}x)\right) = 0, \frac{\pi}{2}, \pi \qquad \boxed{0 \le \cos^{-1}\theta \le \pi}$$

$$\cot(2\tan^{-1}x) = \cos 0, \cos\frac{\pi}{2}, \cos\pi$$

$$= 1, 0, -1$$

$$2\tan^{-1}x = \frac{\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{4}, -\frac{\pi}{4}, \ldots$$

$$\tan^{-1}x = \frac{\pi}{8}, -\frac{3\pi}{8}, \frac{\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{8}, -\frac{\pi}{8} \qquad \boxed{-\frac{\pi}{2} < \tan^{-1}\theta < \frac{\pi}{2}}$$

$$x = \tan\frac{\pi}{8}, \tan\left(-\frac{3\pi}{8}\right), \tan\frac{\pi}{4}, \tan\left(-\frac{\pi}{4}\right), \tan\frac{3\pi}{8}, \tan\left(-\frac{\pi}{8}\right)$$

$$x = \sqrt{2}-1, -\sqrt{2}-1, 1, -1, \sqrt{2}+1, -\sqrt{2}+1$$

METHOD 2

let $\alpha = \tan^{-1}x \quad , -\frac{\pi}{2} < \alpha < \frac{\pi}{2}$

$\tan\alpha = x$

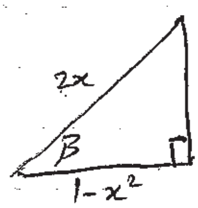
$$\cot(2\alpha) = \frac{1}{\tan 2\alpha}$$

$$= \frac{1}{\left(\frac{2\tan\alpha}{1-\tan^2\alpha}\right)}$$

$$= \frac{1-x^2}{2x}$$

let $\beta = \cos^{-1}\left(\frac{1-x^2}{2x}\right) \quad , 0 \le \beta \le \pi$

$$\cos\beta = \frac{1-x^2}{2x}$$



$$\sqrt{(2x)^2-(1-x^2)^2}$$

$$= \sqrt{4x^2-(1-2x^2+x^4)}$$

$$= \sqrt{6x^2-x^4-1}$$

$$\sin 2\beta = 2\sin\beta\cos\beta$$

$$\sin 2\beta = 2 \cdot \frac{\sqrt{6x^2-x^4-1}}{2x} \cdot \frac{1-x^2}{2x}$$

$$= \frac{2(1-x)(1+x)\sqrt{6x^2-x^4-1}}{4x^2}$$

$$\sin 2\beta = 0$$

$$\frac{2(1-x)(1+x)\sqrt{6x^2-x^4-1}}{4x^2} = 0$$

$x = \pm 1$ or

$$6x^2 - x^4 - 1 = 0$$
$$x^4 - 6x^2 + 9 = -1 + 9$$
$$(x^2-3)^2 = 8$$
$$x^2 - 3 = \pm 2\sqrt{2}$$
$$x^2 = 3 \pm 2\sqrt{2}$$
$$x = \pm\sqrt{3 \pm 2\sqrt{2}}$$

Note: $\sqrt{3+2\sqrt{2}} = 1+\sqrt{2}$

$\sqrt{3-2\sqrt{2}} = \sqrt{2}-1$

COMMENT:

Students found more success using METHOD 1 as long as they considered enough values for $\sin\theta = 0$

The algebra encountered in METHOD 2 meant many students stopped short of finding the irrational solutions.

d) i) $\frac{m}{m+1}$ (m must be first)

ii) $\frac{m}{m+2} \times \frac{m-1}{m+1} + \frac{m}{m+2} \times \frac{2}{m+1} \times \frac{m-1}{m}$ (m first & second or first & third)

$$= \frac{\cancel{m}(m-1)}{\cancel{(m+2)}(m+1)\cancel{m}}\cancel{(m+2)}$$

$$= \frac{m-1}{m+1}$$

COMMENT:

Students who didn't use factorials had much more success in simplifying the algebra.

The complement could also have been used.

i) $1 - \frac{1}{m+1}$

$= \frac{m+1-1}{m+1}$

$= \frac{m}{m+1}$

ii) $1 - \left(\frac{2}{m+2} + \frac{m}{m+2} \times \frac{2}{m+1} \times \frac{1}{m}\right)$

$= 1 - \frac{2}{(m+2)(m+1)m}\left(m(m+1) + m\right)$

$= 1 - \frac{2}{(m+2)(m+1)m}\left(m^2 + m + m\right)$

$= 1 - \frac{2}{(m+2)(m+1)m}\left(m^2 + 2m\right)$

$= 1 - \frac{2}{(m+2)(m+1)m}\left(m(m+2)\right)$

$= 1 - \frac{2}{m+1}$

$= \frac{m+1-2}{m+1}$

$= \frac{m-1}{m+1}$

Question 15

(a) $P(z)=z^4-2z^3-z^2+2z+10$

$z-(2-i)$ is a factor.

$\therefore 2-i$ is a root.

$\therefore 2+i$ is a root (Conjugate Root Theorem)

$\therefore z-(2+i)$ is a factor, and so is

$$(z-(2-i))(z-(2+i)).$$
$$=z^2-z((2+i)+(2-i))+(2+i)(2-i).$$
$$=z^2-4z+5$$

By long division:

$$\begin{array}{r|l} & z^2+2z+2 \\ z^2-4z+5 & z^4-2z^3-z^2+2z+10 \\ & z^4-4z^3+5z^2 \\ & \quad 2z^3-6z^2+2z \\ & \quad 2z^3-8z^2+10z \\ & \qquad 2z^2-8z+10 \\ & \qquad 2z^2-8z+10 \\ & \qquad 0 \end{array}$$

$$\therefore P(z)=(z^2-4z+5)(z^2+2z+2)$$

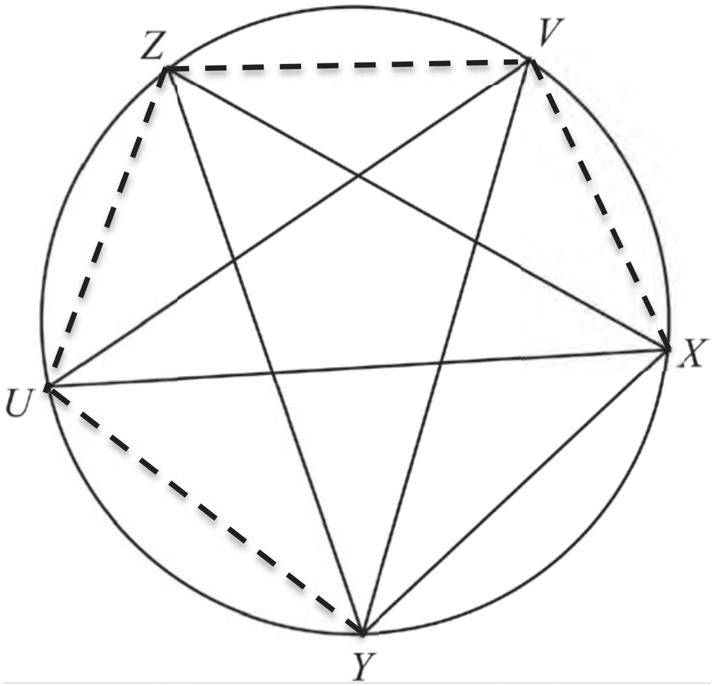
Comment: There is some confusion as to what is a factor and what is a root.

Having established the conjugate root, most successful candidates used long division. Many found the other roots by inspection, or by the factor theorem, but several failed in the attempt.

(b) The 13^{th} disk chosen is the 10^{th} disk with data.
Thus the 3 blank disks have already been selected.
Hence the last three are data disks.
The probability of this is the same as the first three being data disks.

$$p=\frac{12}{15}\times\frac{11}{14}\times\frac{10}{13}$$
$$=\frac{44}{91}$$

(c)



Let $\angle UXY = \angle UXZ = \alpha$
Let $\angle VYX = \angle VYZ = \beta$

Now $\angle UYZ = \angle UXZ = \alpha$ (Standing on same arc)
And $\angle VXZ = \angle VYZ = \beta$ (ditto)

In $\triangle ZYX$, let $\angle YZX = \gamma$

$$\gamma = 180^\circ - 2(\alpha + \beta) \text{ (angle sum of triangle)}$$

Now since *XY* is fixed, so is the angle γ (angle at the circumference).

$$\therefore\ 2(\alpha + \beta) = 180^\circ - \gamma \quad \text{(constant)}$$

$$(\alpha + \beta) = 90^\circ - \frac{\gamma}{2} \quad \text{(constant)}$$

But *UV* subtends $\alpha + \beta$ at *X*, and at *Y*.
Since the angle is constant, so *UV* is of constant length.

Comment: Very few seemed to be aware of the theorem used in this solution.
Of those who attempted a solution, many tried to used similar triangles, but there is no reason for the enlargement ratio to be constant. They were given half marks.

(d) (i) $z = r(\cos\theta + i\sin\theta)$

$$z - \overline{z} = r(\cos\theta + i\sin\theta) - r(\cos\theta - i\sin\theta)$$
$$= r(2i\sin\theta)$$
$$= 2ir\sin\theta$$

Comment: Most candidates got this question right.

(ii) (1)

$$x^m(1-x)^n = A(x)Q(x) + R(x)$$
$$x^m(1-x)^n = (1+x^2)Q(x) + (ax+b)$$

The division transformation is an identity.
That is, it is true for all values of x.

Let $x = i$ $\quad i^m(1-i)^n = (1-i^2)Q(i) + (ai+b)$

$\therefore i^m(1-i)^n = ai + b \quad Eqn(1)$

Let $x = -i$ $\quad i^m(1+i)^n = (1-(-i)^2)Q(i) - ai + b$

$\therefore i^m(1+i)^n = -ai + b \quad Eqn(2)$

(1)-(2): $\quad 2ai = i^m(1-i)^n - (-i)^m(1+i)^n$

Comment: Few candidates attempted this question, about half of whom got it right. The rest chose their substituted values poorly.

(2) Note that $\overline{i^m(1-i)^n} = (-i)^m(1+i)^n$

Thus from above $2ai = 2ir\sin\theta$

Now $\arg\left(i^m\right) = \dfrac{m\pi}{2}$

$\arg(1-i)^n = -\dfrac{n\pi}{4}$

$$\arg\left(i^m\right)(1-i)^n = \frac{m\pi}{2} - \frac{n\pi}{4}$$
$$= \frac{(2m-n)\pi}{4}$$

Now $\left|i^m(1-i)^n\right| = \left|i^m\right| \times \left|(1-i)^n\right|$

$= 1 \times \left(\sqrt{2}\right)^n$

Hence $a = \left(\sqrt{2}\right)^n \sin\dfrac{(2m-n)\pi}{4}$

Comment: Very few attempted this, of whom some were successful, having noticed the conjugate relationship.

Question 15

a) $I = \int_0^a \frac{\cos x}{\sin x + \cos x}\,dx \qquad J = \int_0^a \frac{\sin x}{\sin x + \cos x}\,dx$

where $0 \le a \le \frac{3\pi}{4}$

$$I + J = \int_0^a \frac{\sin x + \cos x}{\sin x + \cos x}\,dx$$
$$= \int_0^a dx$$
$$= [x]_0^a$$
$$= a - 0$$
$$I + J = a \quad \text{——— (1)}$$

$$I - J = \int_0^a \frac{\cos x - \sin x}{\sin x + \cos x}\,dx$$
$$= \left[\ln(\sin x + \cos x)\right]_0^a$$
$$= \ln(\sin a + \cos a) - \ln(\sin 0 + \cos 0)$$
$$I - J = \ln(\sin a + \cos a) \quad \text{——— (2)}$$

(1) + (2)

$$2I = a + \ln(\sin a + \cos a)$$

COMMENT

Some students were looking to apply the result $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ which would be useful when $a = \frac{\pi}{2}$.
That would involve ignoring J. Students who did ignore J scored poorly.

b) i) $I = \int \frac{dx}{\sqrt{x(x+1)}}$

$x = \frac{1}{t^2-1}$

$x = (t^2-1)^{-1}$

$\frac{dx}{dt} = -(t^2-1)^{-2} \cdot 2t$

$dx = -\frac{2t}{(t^2-1)^2}$

$$I = \int \frac{1}{\sqrt{\left(\frac{1}{t^2-1}\right)\left(\frac{1}{t^2-1}+1\right)}} \cdot \frac{-2t\,dt}{(t^2-1)^2}$$

$$= \int \frac{-2t\,dt}{(t^2-1)^2\sqrt{\frac{1+t^2-1}{(t^2-1)^2}}}$$

$$= \int \frac{-2t\,dt}{(t^2-1)^2\sqrt{\frac{t^2}{(t^2-1)^2}}}$$

$$= \int \frac{-2t\,dt}{(t^2-1)^2 \cdot \frac{t}{t^2-1}}$$

since $t > 1$, $\sqrt{t^2} = t$, $\sqrt{(t^2-1)^2} = (t^2-1)$

$$= \int \frac{-2\,dt}{t^2-1}$$

PARTIAL FRACTIONS

$\frac{-2}{t^2-1} \equiv \frac{A}{t-1} + \frac{B}{t+1}$

$-2 \equiv A(t+1) + B(t-1)$

when $t=1$

$-2 = 2A$

$A = -1$

when $t=-1$

$-2 = -2B$

$B = 1$

$\frac{-2}{t^2-1} \equiv -\frac{1}{t-1} + \frac{1}{t+1}$

$$I = \int \left(-\frac{1}{t-1} + \frac{1}{t+1}\right) dt$$

$$= -\ln(t-1) + \ln(t+1) + C$$

$$= \ln\left(\frac{t+1}{t-1}\right) + C$$

$$x = \frac{1}{t^2-1}$$

$$t^2 - 1 = \frac{1}{x}$$

$$t^2 = \frac{1}{x} + 1$$

$$t^2 = \frac{x+1}{x}$$

$$t = \frac{\sqrt{x+1}}{\sqrt{x}}$$

$$= \ln\left(\frac{\frac{\sqrt{x+1}}{\sqrt{x}} + 1}{\frac{\sqrt{x+1}}{\sqrt{x}} - 1} \times \frac{\sqrt{x}}{\sqrt{x}}\right) + C$$

$$= \ln\left(\frac{\sqrt{x+1} + \sqrt{x}}{\sqrt{x+1} - \sqrt{x}} \times \frac{\sqrt{x+1} + \sqrt{x}}{\sqrt{x+1} + \sqrt{x}}\right) + C$$

$$= \ln\left(\frac{(\sqrt{x+1} + \sqrt{x})^2}{x+1-x}\right) + C$$

$$= 2\ln\left(\sqrt{x} + \sqrt{x+1}\right) + C$$

ii) $$\int_{\frac{1}{8}}^{\frac{9}{16}} \left(\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{x+1}}\right)^2 dx$$

$$= \int_{\frac{1}{8}}^{\frac{9}{16}} \left(\frac{1}{x} - \frac{2}{\sqrt{x(x+1)}} + \frac{1}{x+1}\right) dx$$

$$= \left[\ln x - 2\left(2\ln\left(\sqrt{x} + \sqrt{x+1}\right)\right) + \ln(x+1)\right]_{\frac{1}{8}}^{\frac{9}{16}}$$

$$= \ln\left(\tfrac{9}{16}\right) - 4\ln\left(\sqrt{\tfrac{9}{16}} + \sqrt{\tfrac{9}{16}+1}\right) + \ln\left(\tfrac{9}{16}+1\right) - \left(\ln\left(\tfrac{1}{8}\right) - 4\ln\left(\sqrt{\tfrac{1}{8}} + \sqrt{\tfrac{1}{8}+1}\right) + \ln\left(\tfrac{1}{8}+1\right)\right)$$

$$= \ln\left(\frac{\frac{9}{16} \times \frac{25}{16}}{\frac{1}{8} \times \frac{9}{8}}\right) + 4\ln\left(\frac{\frac{1}{2\sqrt{2}} + \frac{3}{2\sqrt{2}}}{\frac{3}{4} + \frac{5}{4}}\right)$$

$$= \ln\left(\frac{25}{4}\right) + 4\ln\left(\frac{\frac{4}{2\sqrt{2}}}{2}\right)$$

$$= \ln\left(\frac{5}{2}\right)^2 + 4\ln\left(\frac{1}{\sqrt{2}}\right)$$

$= 2\ln\left(\frac{5}{2}\right) + 2\ln\left(\frac{1}{\sqrt{2}}\right)^2$

$= 2\ln\left(\frac{5}{2}\right) + 2\ln\left(\frac{1}{2}\right)$

$= 2\ln\left(\frac{5}{2} \times \frac{1}{2}\right)$

$= 2\ln\left(\frac{5}{4}\right)$

COMMENTS:

i) There is a lot of algebra. Since the question asks us to SHOW THAT much care needs to be taken and all steps shown ie. when rationalising the denominator inside the logarithm

Note: If we did not have to use the substitution given it would be quite simple to complete the square on $x(x+1)$ and use a standard integral.

ii) Again, as this question was a SHOW THAT students needed to show how the expression simplified to be awarded full marks.

Some students struggled to see how part (i) could be used.

c) i) $I = \int [f'(x)]^2 [f(x)]^n \, dx$

$= \int f'(x) . f'(x) [f(x)]^n \, dx$

$u = f'(x) \qquad v' = f'(x)[f(x)]^n$

$u' = f''(x) \qquad v = \frac{[f(x)]^{n+1}}{n+1}$

$I = \frac{f'(x)[f(x)]^{n+1}}{n+1} - \frac{1}{n+1}\int f''(x)[f(x)]^{n+1} dx + C_1$

$I = \frac{f'(x)[f(x)]^{n+1}}{n+1} - \frac{1}{n+1}\int k f'(x) f(x) [f(x)]^{n+1} dx + C_1$

$$I = \frac{f'(x)[f(x)]^{n+1}}{n+1} - \frac{k}{n+1}\int f'(x)[f(x)]^{n+2}\,dx + C_1$$

$$= \frac{f'(x)[f(x)]^{n+1}}{n+1} - \frac{k}{(n+1)} \cdot \frac{[f(x)]^{n+3}}{(n+3)} + C_2$$

$$= \frac{f'(x)[f(x)]^{n+1}}{n+1} - \frac{k[f(x)]^{n+3}}{(n+1)(n+3)} + C_2$$

ii) $f(x) = \sec x + \tan x$

$f'(x) = \sec x \tan x + \sec^2 x$

$= \sec x(\sec x + \tan x)$

$f''(x) = \sec x \tan x(\sec x + \tan x) + \sec x(\sec x \tan x + \sec^2 x)$

$= \sec^2 x \tan x + \sec x \tan^2 x + \sec^2 x \tan x + \sec^3 x$

$= \sec^3 x + 2\sec^2 x \tan x + \sec x \tan^2 x$

$k f'(x) f(x) = k[\sec x(\sec x + \tan x)](\sec x + \tan x)$

$= k \sec x(\sec x + \tan x)^2$

$= k \sec x(\sec^2 x + 2\sec x \tan x + \tan^2 x)$

$= k[\sec^3 x + 2\sec^2 x \tan x + \sec x \tan^2 x]$

$\therefore k = 1.$

$[f'(x)]^2 = \sec^2 x(\sec x + \tan x)^2$

$\int \sec^2 x(\sec x + \tan x)^6\,dx$

$= \int \sec^2 x(\sec x + \tan x)^2 \cdot (\sec x + \tan x)^4\,dx$

$$= \frac{\sec x(\sec x + \tan x)^6}{5} - \frac{1.(\sec x + \tan x)^7}{5 \times 7} + C$$

$$= \frac{\sec x(\sec x+\tan x)^6}{5} - \frac{(\sec x+\tan x)^7}{35} + C$$

$$= \frac{(\sec x+\tan x)^6}{35}\left(7\sec x - (\sec x+\tan x)\right) + C$$

$$= \frac{(\sec x+\tan x)^6(6\sec x - \tan x)}{35} + C$$

COMMENTS:

i) Many students tried to integrate $[f(x)]^n$ instead of $f'(x)[f(x)]^n$.

ii) No student checked if $f''(x) = kf'(x)f(x)$ could be satisfied.
If $[f'(x)]^2$ was not considered the wrong value of n was chosen.

Overall, there were very few attempts at this question.